We proved the following theorem:

Thm Let $F: S \to S'$ be a homomorphism with kernel $K \subset S$.

Let $N \subset S$ be a normal subgroup that is *contained* in K.

Then there exists a *unique* homomorphism

$$h: S/N \to S'$$

s.t. the following diagram is commutative:

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr] & & S' \\ & S/N \arrow[ur, "h"'] & \end{tikzcd}$$

Let me recall how we construct the homomorphism h.

(1) We identify $S \xrightarrow{f} S'$ with

$$\begin{array}{ccccc} S & \twoheadrightarrow & \operatorname{im} f & \hookrightarrow & S' \\ \in & & \in & & \in \\ g & \longmapsto & f(g) & \longmapsto & f(g) \end{array}$$

(2) We use main theorem of Lecture 9 to identify

$$\begin{array}{ccc} S/K & \xrightarrow{\simeq} & \operatorname{im} f \\ \in & & \in \\ gK & \longmapsto & f(g) \end{array}$$

So, we can identify f with:

$$
\begin{tikzcd}
S \arrow[r, two heads] \arrow[rr, bend left, "f"] & S/K \arrow[r, hook] & S' \\
g \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gK \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & f(g) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}
$$

Now, we present
$$
\begin{tikzcd}
S \arrow[r, two heads] & S/K \\
g \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gk \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}
$$
as a composition:

$$
\begin{tikzcd}
S \arrow[r, two heads, "\varphi"] & S/N \arrow[r, two heads, "\pi"] & S/K \\
g \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gN \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gK
\end{tikzcd}
$$

We claim that this map is well-defined and defines a <u>homomorphism</u> of groups

<u>Indeed:</u> it is well-defined because if we replace g by gn for some $n \in N$ then $gnN \longmapsto gnK = gK$

use that $n \in N \subset K$

This map is indeed a homomorphism:

$$\pi(g_1N \cdot g_2N) = \pi(g_1g_2N) =$$

$$= g_1g_2K = g_1K \cdot g_2K = \pi(g_1N) \cdot \pi(g_2N) \checkmark$$

So, we obtain the decomposition of f:

$$\begin{tikzcd}
S \arrow[r, twoheadrightarrow, "\varphi"] \arrow[rrr, bend left, "f"] & S/N \arrow[r, twoheadrightarrow, "\pi"] & S/K \arrow[r, hookrightarrow, "j"] & S'
\end{tikzcd} \quad (*)$$

$$g \longmapsto gN \longmapsto gK \longmapsto f(g)$$

Let $j : S/K \hookrightarrow S'$ be the embedding

$$\begin{aligned} j : S/K &\hookrightarrow S' \\ gK &\longmapsto f(g) \end{aligned}$$

We define $h := j \circ \pi : S/N \to S'$

Note that the diagram:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "\varphi"'] & S' \\ & S/N \arrow[u, "h"'] \end{tikzcd}$$

is indeed commutative:

$$h \circ \varphi = j \circ \pi \circ \varphi = f \quad \text{by } (*) \text{ above}$$